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Derivatives of Polynomials with Positive Coefficients[1]

G. G. LORENTZ

Department of Mathematics, Syracuse University, Syracuse, New York 13210

The paper [*1*] contains the following result. There exists a constant $C > 0$ such that for each polynomial of the form $P_n(x) = \sum_0^n b_k x^k(1-x)^{n-k}$, $b_k \geqslant 0$,

$$\|P_n'\|/\|P_n\| \leqslant Cn, \qquad n = 1, 2, \ldots, \tag{1}$$

for the uniform norm on [0, 1]. This relation can also be written

$$\frac{\|P_n'\|}{\|P_n\|} \leqslant C \frac{\|p_n'\|}{\|p_n\|}, \tag{2}$$

where p_n are the special polynomials $p_n(x) = x^n$.

In particular, (1) holds for polynomials with positive coefficients in x,

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geqslant 0.$$

This follows from (1), but also immediately, since with P_n, also P_n' is a polynomial with positive coefficients, and since for such P_n, $\|P_n\| = P_n(1)$. In the present note we prove the inequality (2) for the infinite interval $(0, +\infty)$, and for a supremum norm with weight. The norm of a function f on $(0, +\infty)$ is given by

$$\|f\| = \sup_{x \geqslant 0} |f(x)\, e^{-\omega(x)}|, \tag{3}$$

where ω increases on $(0, +\infty)$. In addition to some mild smoothness requirements for ω, we shall assume that ω does not increase too slowly. Thus, Theorem 2 allows $\omega(x) = \log^p x$, $p > 1$, but functions $\omega(x) = A \log x$ are, of course, excluded.

In what follows, we shall assume that $\omega(x)$ is a positive differentiable function, defined for $0 \leqslant x < +\infty$, increasing strictly to $+\infty$, and such that also $x\omega'(x)$ strictly increases to $+\infty$. For each $n = 0, 1, \ldots$, the maximum of $x^n e^{-\omega(x)}$ is attained at a unique point $x = x_n$, given by

$$n = x_n\, \omega'(x_n). \tag{4}$$

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We have $x_n \nearrow +\infty$ as $n \to \infty$. Another remark is that the behavior of $\|p_n'\|/\|p_n\|$ is very much like that of $n/x_n = \omega'(x_n)$:

$$\frac{n}{x_n} \leqslant \frac{\|p_n'\|}{\|p_n\|} \leqslant \frac{n}{x_{n-1}}, \qquad n = 2, 3, \ldots. \tag{5}$$

This follows from the inequalities

$$\|p_n\| = \frac{x_n}{n} n x_n^{n-1} e^{-\omega(x_n)} \leqslant \frac{x_n}{n} \|p_n'\|, \qquad n \geqslant 1,$$

$$\|p_n'\| = \frac{n}{x_{n-1}} x_{n-1}^n e^{-\omega(x_{n-1})} \leqslant \frac{n}{x_{n-1}} \|p_n\|, \qquad n \geqslant 2.$$

The following two theorems deal, roughly, with the cases when $\omega'(x)$ increases and when it decreases.

THEOREM 1. *Let $\omega(x)$ satisfy the inequalities $\omega(x) - \omega(0) \leqslant Ax\omega'(x)$, $x \geqslant 0$, and $\omega'(y) \leqslant A\omega'(x)$, $y \leqslant x$, for some constant $A > 0$. (Both conditions are satisfied if ω' increases or if ω' decreases, but remains bounded away from zero.) Then for some constant $C > 0$, inequality (2) holds for each polynomial P_n with positive coefficients.*

THEOREM 2. *Let $\overline{\lim_{x\to\infty}}\, \omega'(x) < 1$ and assume that for some $0 < q < 1$,*

$$\omega(x) \leqslant qx\omega'(x) \log \frac{1}{\omega'(x)} \text{ for all sufficiently large } x. \tag{6}$$

Then for some constant $C > 0$,

$$\frac{\|P_n'\|}{\|P_n\|} \leqslant C, \qquad n = 1, 2, \ldots \tag{7}$$

Note that if ω' is bounded, then according to (5), $\|p_n'\|/\|p_n\| \leqslant$ Const. The proof depends upon the following

LEMMA. *Let $u_k = \frac{1}{2}(k+1)^{-2}$. There exists a $k_0 \geqslant 1$ for which*

$$\frac{x^k e^{-\omega(x)}}{\|p_k\|} < u_k, \qquad k \geqslant k_0, \tag{8}$$

if x and k satisfy $x < k$ in case of Theorem 2 and $x < cx_k$ in case of Theorem 1 (where $c > 0$ is a constant).

Proof of the Lemma. Assume that (8) is violated for some k. Then

$$k \log(x/x_k) - \omega(x) + \omega(x_k) \geqslant \log u_k. \tag{9}$$

In case of Theorem 1, we have $\omega(x_k) - \omega(x) \leqslant Ax_k\,\omega'(x_k) = Ak$; hence (9) implies

$$k \log (x/x_k) \geqslant -Ak + \log u_k \geqslant -(A+1)k, \qquad k \geqslant k_0,$$

hence $x/x_k \geqslant c$, $c = e^{-(A+1)}$.

Likewise, in the case of Theorem 2, there is, according to (6), a k_0 so that, for $k \geqslant k_0$,

$$\omega(x_k) \leqslant qk \log \frac{1}{\omega'(x_k)} < k \log \frac{1}{\omega'(x_k)} + \log u_k.$$

Therefore, for $k \geqslant k_0$, (9) implies

$$k \log (x/x_k) \geqslant k \log \omega'(x_k)$$

or

$$x \geqslant x_k\,\omega'(x_k) = k.$$

To complete the proof of the theorems, let $S = S(x,n)$ be the set of integers k which satisfy $k_0 < k \leqslant n$ and the inequality $cx_{k-1} \leqslant x$ (in case of Theorem 1) or $k-1 \leqslant x$ (in case of Theorem 2). Let L be the remaining integers k with $k_0 < k \leqslant n$.

For a polynomial with positive coefficients,

$$P_n(t) = \sum_0^n a_k t^k,$$

we put

$$Q_n(t) = \sum_{k_0 < k \leqslant n} a_k t^k.$$

We can assume that $\|Q_n'\| > 0$, for if $Q_n(t)$ vanishes, the following proof is simplified. Let x be such that $Q_n'(x) = \|Q_n'\|$ (obviously, $x_1 \leqslant x \leqslant x_n$). Then

$$\|P_n'\| \leqslant \sum_{k \leqslant k_0} ka_k \|p_{k-1}\| + \|Q_n'\| = \Sigma_1 + \|Q_n'\|; \tag{10}$$

$$\|Q_n'\| = \sum_{k \in L} ka_k x^{k-1} e^{-\omega(x)} + \sum_{k \in S} = \Sigma_2 + \Sigma_3,$$

say. With $M = \max\limits_{k \leqslant k_0} (k\|p_{k-1}\|/\|p_k\|)$,

$$\Sigma_1 \leqslant M \sum_{k \leqslant k_0} \|a_k p_k\| \leqslant (k_0+1)M\|P_n\|. \tag{11}$$

For $k \in L$, we have (8) with k replaced by $k-1$. Therefore

$$\Sigma_2 \leqslant \sum_{k_0 < k \leqslant n} ka_k \|p_{k-1}\| \frac{1}{2k^2} \leqslant \|Q_n'\| \sum_{k=1}^{\infty} \frac{1}{2k^2} = (1-a)\|Q_n'\|,$$

where a, $0 < a < 1$, is an absolute constant. This implies that

$$\Sigma_3 \geqslant a\|Q_n'\|. \tag{12}$$

Our last computation is different in the cases of Theorem 1 and Theorem 2. We note that

$$\|P_n\| \geqslant \sum_{k\in S} \frac{x}{k} ka_k x^{k-1} e^{-\omega(x)}.$$

In the first case, with $c_1 = ac$,

$$\|P_n\| \geqslant c \sum_{k\in S} \frac{x_{k-1}}{k} ka_k x^{k-1} e^{-\omega(x)} \geqslant c_1 \|Q_n'\| \min_{1\leqslant k\leqslant n} \frac{1}{\omega'(x_k)}.$$

Hence, by the assumptions of Theorem 1 and (5),

$$\|Q_n'\| \leqslant \frac{1}{c_1} \max_{1\leqslant k\leqslant n} \omega'(x_k) \|P_n\|$$

$$\leqslant \frac{A}{c_1} \omega'(x_n) \|P_n\| \leqslant \frac{A}{c_1} \|P_n\| \frac{\|p_n'\|}{\|p_n\|}. \tag{13}$$

From (10), (11), and (13) we obtain (2), since $\|p_n'\|/\|p_n\|$ bounded from below.

In the second case, since $x/k > \frac{1}{2}$ for $x \in S$,

$$\|P_n\| \geqslant \frac{1}{2} \sum_{k\in S} ka_k x^{k-1} e^{-\omega(x)} = \tfrac{1}{2}\Sigma_3 \geqslant \frac{a}{2} \|Q_n'\|, \tag{14}$$

and we obtain (7) from (10), (11), and (14).

We make some additional remarks. In [*3*], Szegö studied the order of magnitude of $\|P_n'\|/\|P_n\|$ for unrestricted polynomials P_n, for the norm $\|f\| = \sup_{x\geqslant 0} |f(x) e^{-x}|$ on $(0, +\infty)$. He obtained that this does not exceed $C_0 n$. (For the Laguerre polynomials P_n, the quotient is $= n$.) In this case, the largest value of $\|p_n'\|/\|p_n\|$ is $= e$ for $n = 1$; this quotient decreases and has limit 1 for $n \to \infty$. We see that $\|P_n'\|/\|P_n\|$ is much smaller for polynomials with positive coefficients, than in the general case.

It has been found [*2*] that the smallest possible constant C in (1) is $C = e$. Of some interest is the smallest value of Szegö's constant C_0; this has not yet been determined. A possible conjecture is that this, too, is $C_0 = e$.

References

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